3. Probability distributions of discrete random variables

Uniform distribution

In a case, where each of the outcomes of the random experiment is equally probable, the corresponding random variable X, which domain is an interval of integers $[a, b]$, is **uniformly distributed**:

$$f_X(x) = \frac{1}{b-a+1}, \quad x \in [a, b] \subset \mathbb{Z}$$

Properties: $m_X = \frac{a+b}{2}$, $\mathrm{Var}(X) = \frac{(b-a+1)^2-1}{12}$.

Binomial distribution

Bernoulli random experiment results in only two possible results, e.g. "success" and "failure". A random variable X is defined as a number of trials resulting in a "success" while performing n trials of the Bernoulli experiment. The random variable X is **binomial**, if 1) the n trials of the random experiment are independent and 2) the probability p of "success" remains constant while repeating the experiment. Its probability mass function f_X is then a **binomial distribution**:

$$f_X(x) = P(X = x) = \binom{n}{x} p^x (1-p)^{n-x} \quad \text{for} \;\; x = 0,1,2, \dots, n.$$

Properties: $m_X = np$, $\mathrm{Var}(X) = np(1-p)$.

The binomial coefficient is defined by: $\binom{n}{x} = \frac{n!}{x!(n-x)!}$.

Poisson distribution

In case the number of trials n is large and the probability of a "success" p is low, so that $np \sim 1$, the probability mass function of the binomial distribution can be approximated by the probability mass function of the **Poisson distribution** where parameter $\lambda = np$ is used:

$$f_X(x) = P(X = x) = \mathrm{e}^{-\lambda} \frac{\lambda^x}{x!}, \quad x = 0,1,2, \dots$$

Properties: $m_X = \lambda$, $\mathrm{Var}(X) = \lambda$.

The Poisson random variable X can be interpreted as a number of events ("successful outcomes") occurring on some interval (of length l or time t, area A, volume V…). The parameter λ is than a product of mean frequency (rate) ν of event occurrence per unit of the interval (either length or time, area, volume…) and the interval length (either length l or time t, area A, volume V…):

$$\lambda = \nu l \;\; \text{or} \;\; \lambda = \nu t, \lambda = \nu A, \lambda = \nu V.$$

The Poisson distribution is used when 1) the number of event occurrences can be described by an non-negative integer number, 2) the events occur independently, 3) the rate (frequency) at which events occur is constant, 4) two events cannot occur at exactly the same instant, and 5) the probability of an event in a small sub-interval is proportional to the length of the sub-interval.

3. Probability distributions of discrete random variables – problems

1. A gearwheel in a transmission mechanism of a lathe has 33 teeth, which are numbered consecutively from 1 to 33. What is the probability that when the lathe is stopped, the top tooth of the gearwheel has a number between 1 and 12? R: $P = 12/33 = 0.364$

2. Consider shooting at the target, where the probability of hitting the target is 0.3 for every shot. At least how many times should we shoot to hit the target at least once with the probability of 0.9? R: $N = 7$

3. 3% of roof tiles are too porous. We randomly select twenty tiles.
    a. What is the probability that two of them are too porous? R: $P = 0.099$
    b. What is the probability that no more than two of them are too porous? R: $P = 0.980$

4. A system for detecting faults in the material is checked. The system on average correctly detects only every fifth fault. How many faults should there be for the system to detect at least one with a probability of at least 0.95? R: $n = 14$

5. Products are packed in boxes that contain 100 pieces. Among the products there are 0.7 % defective. What is the probability that a randomly selected box holds more than two defective products? R: $P = 0.0336$ (exact) and $P = 0.0341$ (approximation)

6. Number of applications for repairing a product is Poisson distributed with the average of three applications per week.
    a. What is the probability that we get more than four applications in a randomly selected week? R: $P = 0.185$
    b. It is estimated that on average in one third of applications the repair is not needed as the application is a consequence of the user's lack of knowledge. What is the probability that there will be no more than five repairs in a time of four weeks? R: $P = 0.191$

7. It is assumed that the potholes in the road surface are Poisson distributed with the average of three potholes per 20 km of the road.
    a. What is the probability that no more than four potholes are found in the 30 km long road section? R: $P = 0.532$
    b. What is the length of a road section without potholes with the probability of at least 0.9? R: $l = 0.703$ km

8. It is assumed that the number of errors along the drawn wire is Poisson distributed with the average error frequency of 1 per 5 m of the wire.
    a. What is the probability that in 20 m wire length there would be no more than two errors? R: $P = 0.238$
    b. Wire is cut into 1 m long pieces. What is the probability that among the ten randomly selected pieces no more than three will have at least one error? R: $P = 0.910$