1 Statistical distributions

1.1 Sample mean value

The sample mean value of the failure time or an average time to failure in the sample - MTTF (Mean Time To Failure) is an estimate, calculated as

$$MTTF = \frac{1}{n} \cdot \sum_{i=1}^{n} t_i. \tag{1}$$

1.2 Empirical variance

Empirical variance is an estimate calculated as

$$S^2 = \frac{1}{n-1} \cdot \sum_{i=1}^{n} (t_i - MTTF)^2. \tag{2}$$

1.3 Standard deviation

Standard deviation is calculated as

$$S = \sqrt{S^2} \tag{3}$$

and is introduced due to the same units as in $MTTF$.

1.4 Histogram

Histogram is an ordered way of representing the measured data. The data $t_i;\ i = 1, \ldots, n$ is to be sorted into m classes. The number of classes is determined based on the size of the sample n as

$$m = \mathrm{INT}\left(1 + 3.33 \cdot log(n)\right). \tag{4}$$

The size of the class is calculated as

$$\Delta t = \frac{t_{\max} - t_{\min} + \varepsilon}{m}, \tag{5}$$

where $t_{\max}$ and $t_{\min}$ are maximal and minimal failure times in the sample, respectively. ε is an arbitrary small value by which it is ensured that the maximal time $t_{\max}$ is included in the last class.

Lower and upper borders $t_{\mathrm{l}j}$ and $t_{\mathrm{u}j}$ are determined for the classes as well as the middle values of the classes t_j as

$$t_{\mathrm{l}j} = t_{\min} + (j-1) \cdot \Delta t \qquad j = 1, \ldots, m \tag{6}$$

$$t_{\mathrm{u}j} = t_{\min} + j \cdot \Delta t \qquad j = 1, \ldots, m \tag{7}$$

$$t_j = t_{\min} + \left(j - \frac{1}{2}\right) \cdot \Delta t \qquad\qquad j = 1, \ldots, m \tag{8}$$

The corresponding class for each measured data t_i is determined as

$$j_i = \mathrm{INT}\left(\frac{t_i - t_{\min}}{\Delta t}\right) + 1 \qquad\qquad i = 1, \ldots, n \tag{9}$$

Frequency n_j is determined so that for each $j = 1, \ldots, m$ the number of $j_i; i = 1, \ldots, n$ is counted. In Microsoft Excel for example, a function such as COUNTIF*(j_1:j_n, j)* can be utilised.

Relative frequency p_j is calculated as

$$p_j = \frac{n_j}{n}. \tag{10}$$

Finally, normalised relative frequency f_j is calculated as

$$f_j = p_j \cdot \frac{1}{\Delta t} = \frac{n_j}{n \cdot \Delta t}. \tag{11}$$

2 Exponential distribution

2.1 Exponential statistical distribution

Exponential statistical distribution is suitable for description of phenomena where failures of products occur randomly.

Probability density function (PDF):

$$f(t) = \lambda e^{-\lambda t} \tag{12}$$

Cumulative distribution function (CDF):

$$F(t) = \int_0^t f(t)dt = 1 - e^{-\lambda t} \tag{13}$$

Reliability function:

$$R(t) = 1 - F(t) = e^{-\lambda t} \tag{14}$$

Failure rate:

$$\lambda(t) = \lambda \tag{15}$$

3 Normal distribution

3.1 Normal statistical distribution

Normal statistical distribution is suitable for description of phenomena where failures occur due to wear, fatigue or aging of a product.

Probability density function (PDF):

$$f(t) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t-\mu)^2}{2\sigma^2}} \tag{16}$$

Cumulative distribution function (CDF):

$$F(t) = \int_0^t f(t)dt = \Phi\left(\frac{t-\mu}{\sigma}\right) \tag{17}$$

Reliability function:

$$R(t) = 1 - F(t) = 1 - \Phi\left(\frac{t-\mu}{\sigma}\right) \tag{18}$$

Failure rate:

$$\lambda(t) = \frac{\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(t-\mu)^2}{2\sigma^2}}}{1 - \Phi\left(\frac{t-\mu}{\sigma}\right)} \tag{19}$$

4 Weibull distribution

Weibull statistical distribution is suitable for description of phenomena where failures occur either due to early-stage defects, random events, wear, fatigue or aging of a product.

Probability density function (PDF):

$$f(t) = \frac{\beta}{\theta} \cdot \left(\frac{t}{\theta}\right)^{\beta-1} \cdot e^{-(\frac{t}{\theta})^\beta} \tag{20}$$

Cumulative distribution function (CDF):

$$F(t) = \int_0^t f(t^*)dt^* = 1 - e^{-(\frac{t}{\theta})^\beta} \tag{21}$$

Reliability function:

$$R(t) = 1 - F(t) = e^{-(\frac{t}{\theta})^\beta} \tag{22}$$

Failure rate:

$$\lambda(t) = \frac{\beta}{\theta} \cdot \left(\frac{t}{\theta}\right)^{\beta-1} \tag{23}$$

4.1 Determination of parameters β and θ

Parameters β and θ determine the shape and the size of the Weibull statistical distribution. The shape parameter β determines the form of the probability density function $f(t)$.

For the determination of parameters β and θ, the **method of median ranges** can be utilised. The observed failure times $t_i;\ i = 1, \ldots, n$ have to be sorted first in ascending order. Then, each failure time is assigned with a value of a discrete cumulative distribution function F_i':

$$F_i' = \frac{i - 0,3}{n + 0,4}. \tag{24}$$

If logarithm of Eq. 21 for cumulative distribution function $F(t)$ is calculated,

$$1 - F(t) = e^{-(\frac{t}{\theta})^{\beta}}/\ln, \tag{25}$$

then

$$-\ln(1 - F(t) = \frac{t}{\theta}^{\beta}. \tag{26}$$

Another logarithm of Eq. 26 is calculated,

$$-\ln(1 - F(t) = \frac{t}{\theta}^{\beta}/\ln, \tag{27}$$

and

$$\ln(\ln(1 - F(t))^{-1}) = \beta \cdot \ln\frac{t}{\theta}. \tag{28}$$

Eq. 28 can be rewritten as

$$\ln(\ln(1 - F(t))^{-1}) = \beta \cdot \ln(t) - \beta \cdot \ln(\theta). \tag{29}$$

For discrete values of t_i and F_i', Eq. 29 yields

$$\ln(\ln(1 - F_i')^{-1}) = \beta \cdot \ln(t_i) - \beta \cdot \ln(\theta). \tag{30}$$

It can be noticed that Eq. 30 has a form of a linear function $y_i = kx_i + n$. Hence,

$$y_i = \ln(\ln(1 - F_i')^{-1}) \tag{31}$$

$$k = \beta \tag{32}$$

$$x_i = \ln(t_i) \tag{33}$$

$$n = -\beta \cdot \ln(\theta). \tag{34}$$

To determine k and n e.g. Microsoft Excel can be used so that experimental results $\ln(t_i)$ are plotted on the abscissa axis and $\ln(\ln(1 - F_i')^{-1})$ are plotted on the ordinate axis. A linear curve is fitted through these data. If a trend line function is chosen, coefficients k and n are obtained. Knowing the coefficients k and n, the parameters β and θ can be calculated.