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Katedra za strojne elemente
in razvojna vrednotenja

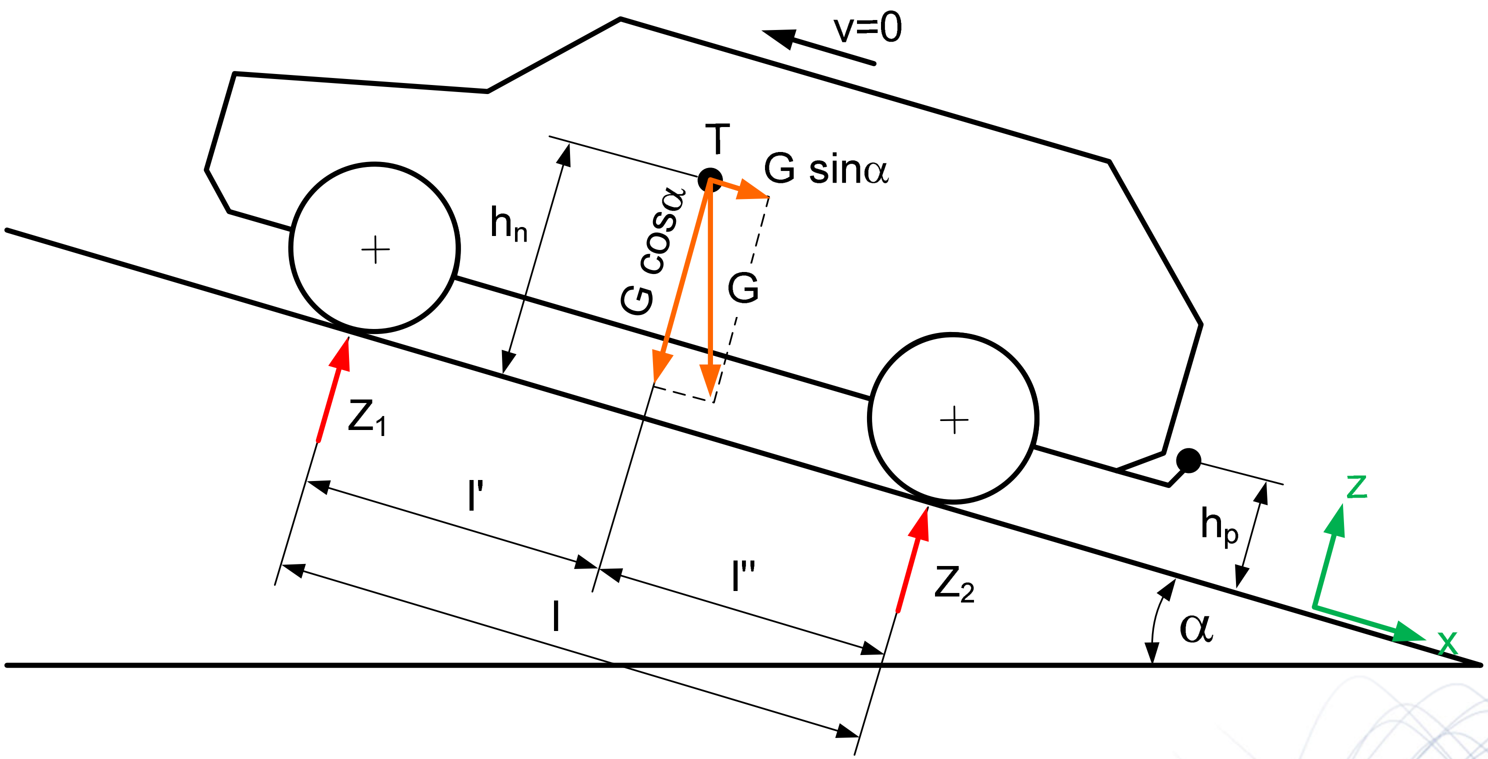
LAVEK
LABORATORIJ ZA VREDNOTENJE KONSTRUKCIJ

LASEM
LABORATORIJ ZA STROJNE ELEMENTE

Balance of forces that act on a vehicle

assist. prof. Simon Oman

Static force balance on a hill



$$Z_1 = \frac{l''}{l} \cdot G \cdot \cos\alpha - \frac{h_n}{l} \cdot G \cdot \sin\alpha$$

$$Z_2 = \frac{l'}{l} \cdot G \cdot \cos\alpha + \frac{h_n}{l} \cdot G \cdot \sin\alpha$$

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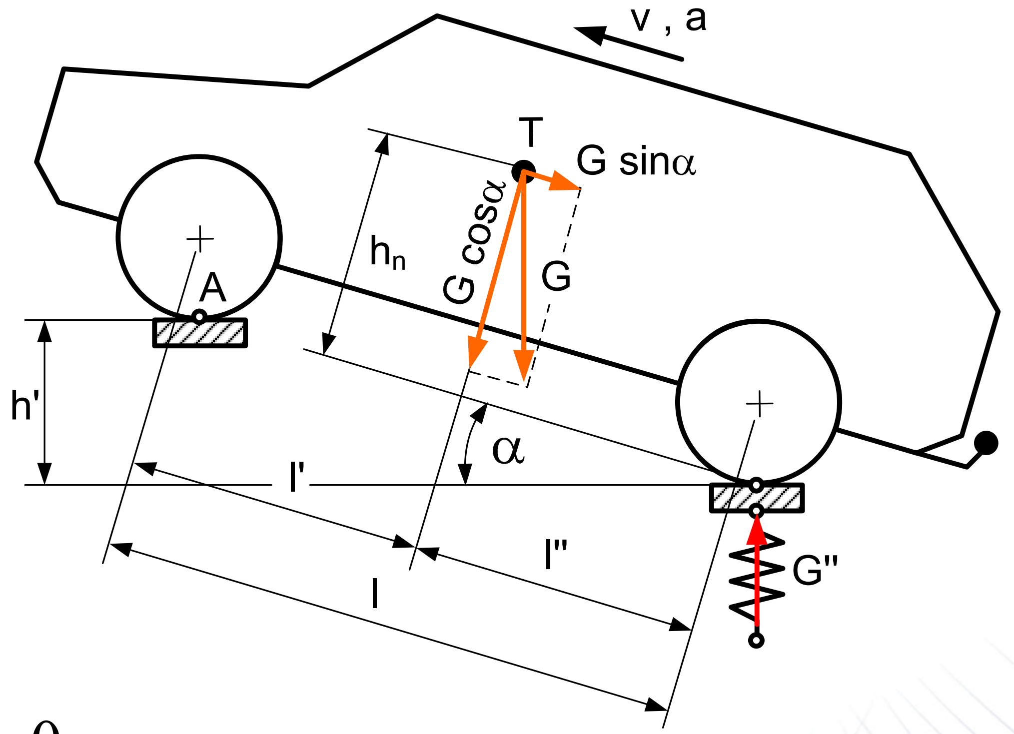
Static force balance on a hill

Estimating a center-of-gravity height:



$$\sum M_A = 0:$$

$$l \cdot G'' \cdot \cos\alpha - l' \cdot G \cdot \cos\alpha - h_n \cdot G \cdot \sin\alpha = 0$$

$$h_n = \left(\frac{G''}{G} \cdot l - l' \right) \cdot \operatorname{ctg}\alpha$$

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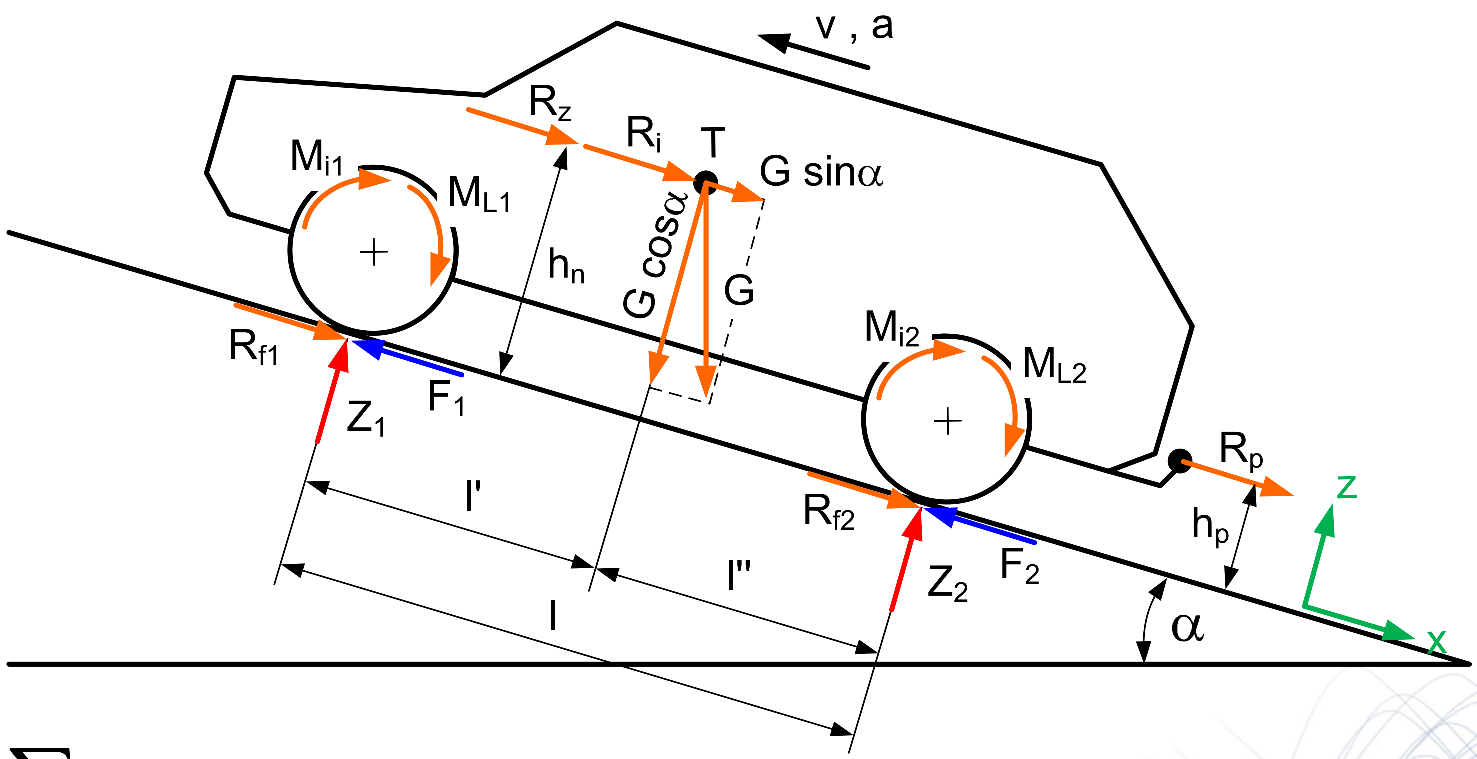
Dynamic force balance on a hill during acceleration



$$\sum M_{(2)} = 0 :$$

$$Z_1 \cdot l = l'' \cdot G \cdot \cos\alpha - h_n \cdot (G \cdot \sin\alpha + R_i + R_z) -$$

$$-(h_p \cdot R_p) - \underbrace{M_{i1} - M_{i2}}_{\approx 0} - M_{L1} - M_{L2}$$

Dynamic force balance on a hill during acceleration

$$\sum F_x = 0$$

$$F_1 + F_2 - R_{f1} - R_{f2} = G \cdot \sin\alpha + R_i + R_z + (R_p)$$

$$F_1 + F_2 = F; R_{f1} + R_{f2} = R_f = f \cdot G \cdot \cos\alpha; R_p = 0$$

$$Z_1 = G \cdot \cos\alpha \cdot \left(\frac{l''}{l} + \frac{f \cdot h_n}{l} \right) - \frac{h_n}{l} \cdot F - \frac{M_{L1} + M_{L2}}{l}$$

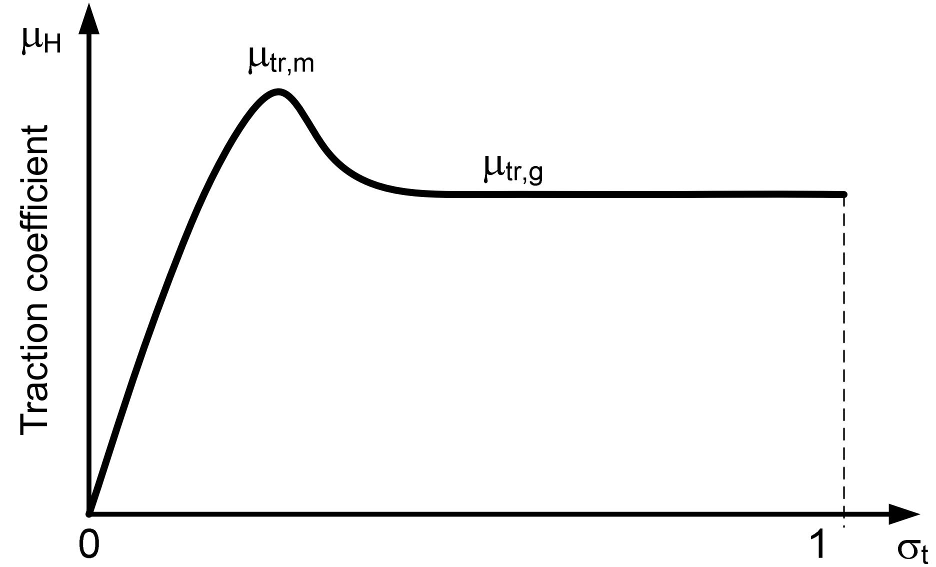
$$\frac{Z_1}{G} = \cos\alpha \cdot \left(\frac{l''}{l} + \frac{f \cdot h_n}{l} \right) - \frac{h_n}{l \cdot G} \cdot F - \frac{M_{L1} + M_{L2}}{l \cdot G}$$

$$Z_2 = G \cdot \cos\alpha \cdot \left(\frac{l'}{l} - \frac{f \cdot h_n}{l} \right) + \frac{h_n}{l} \cdot F + \frac{M_{L1} + M_{L2}}{l}$$

$$\frac{Z_2}{G} = \cos\alpha \cdot \left(\frac{l'}{l} - \frac{f \cdot h_n}{l} \right) + \frac{h_n}{l \cdot G} \cdot F + \frac{M_{L1} + M_{L2}}{l \cdot G}$$

Maximum traction forces

Front-wheel drive:



$$F_{1,\max} = Z_1 \cdot \mu_{H,\max} = Z_1 \cdot \mu_{tr,m}$$

$$F_{1,\max} = \mu_{tr,m} \cdot \left[\frac{l''}{l} \cdot G \cdot \cos\alpha - \frac{h_n}{l} \cdot (F_{1,\max} - f \cdot G \cdot \cos\alpha) \right]$$

$$\frac{F_{1,\max}}{G} = \mu_{tr,m} \cdot \cos\alpha \cdot \frac{l'' + h_n \cdot f}{l + h_n \cdot \mu_{tr,m}}$$

Maximum traction forces

Rear-wheel drive:

$$F_{2,\max} = Z_2 \cdot \mu_{H,\max} = Z_2 \cdot \mu_{tr,m}$$

$$F_{2,\max} = \mu_{tr,m} \cdot \left[\frac{l'}{l} \cdot G \cdot \cos\alpha + \frac{h_n}{l} \cdot (F_{2,\max} - f \cdot G \cdot \cos\alpha) \right]$$

$$\frac{F_{2,\max}}{G} = \mu_{tr,m} \cdot \cos\alpha \cdot \frac{l' - h_n \cdot f}{l - h_n \cdot \mu_{tr,m}}$$

All-wheel drive:

$$\frac{F_{4x4,\max}}{G} = \mu_{tr,m} \cdot \cos\alpha$$

$$\frac{F_{4x4,1}}{F_{4x4,2}} = \frac{Z_1}{Z_2}$$

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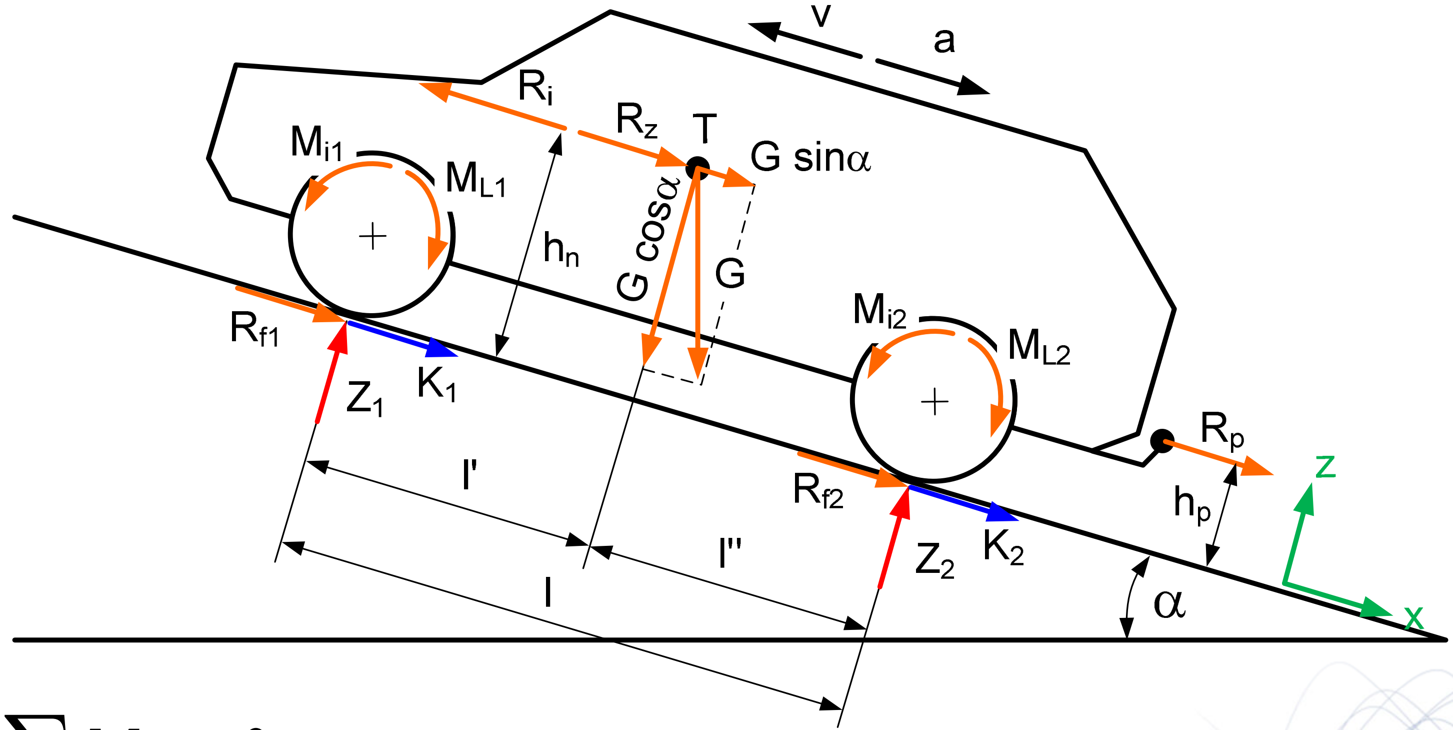
Dynamic force balance on a hill during braking



$$\sum M_{(2)} = 0:$$

$$Z_1 \cdot l = l'' \cdot G \cdot \cos\alpha + h_n \cdot (R_i - G \cdot \sin\alpha - R_z) -$$

$$-(h_p \cdot R_p) + \underbrace{M_{i1} + M_{i2}}_{\approx 0} - M_{L1} - M_{L2}$$

Dynamic force balance on a hill during braking

$$\sum F_x = 0$$

$$K_1 + K_2 + R_{f1} + R_{f2} = R_i - \left[G \cdot \sin\alpha + R_z + (R_p) \right]$$

$$K_1 + K_2 = K; R_{f1} + R_{f2} = R_f = f \cdot G \cdot \cos\alpha; R_p = 0$$

$$Z_1 = G \cdot \cos\alpha \cdot \left(\frac{l''}{l} + \frac{f \cdot h_n}{l} \right) + \frac{h_n}{l} \cdot K - \frac{M_{L1} + M_{L2}}{l}$$

$$\frac{Z_1}{G} = \cos\alpha \cdot \left(\frac{l''}{l} + \frac{f \cdot h_n}{l} \right) + \frac{h_n}{l \cdot G} \cdot K - \frac{M_{L1} + M_{L2}}{l \cdot G}$$

$$Z_2 = G \cdot \cos\alpha \cdot \left(\frac{l'}{l} - \frac{f \cdot h_n}{l} \right) - \frac{h_n}{l} \cdot K + \frac{M_{L1} + M_{L2}}{l}$$

$$\frac{Z_2}{G} = \cos\alpha \cdot \left(\frac{l'}{l} - \frac{f \cdot h_n}{l} \right) - \frac{h_n}{l \cdot G} \cdot K + \frac{M_{L1} + M_{L2}}{l \cdot G}$$

Maximal braking forces

$$M_{L1} + M_{L2} \approx 0$$

$$K_1 = Z_1 \cdot \mu_H$$

$$K_2 = Z_2 \cdot \mu_H$$

$$K_1 + K_2 = K = Z_1 \cdot \mu_H + Z_2 \cdot \mu_H = \mu_H \cdot G \cdot \cos\alpha$$

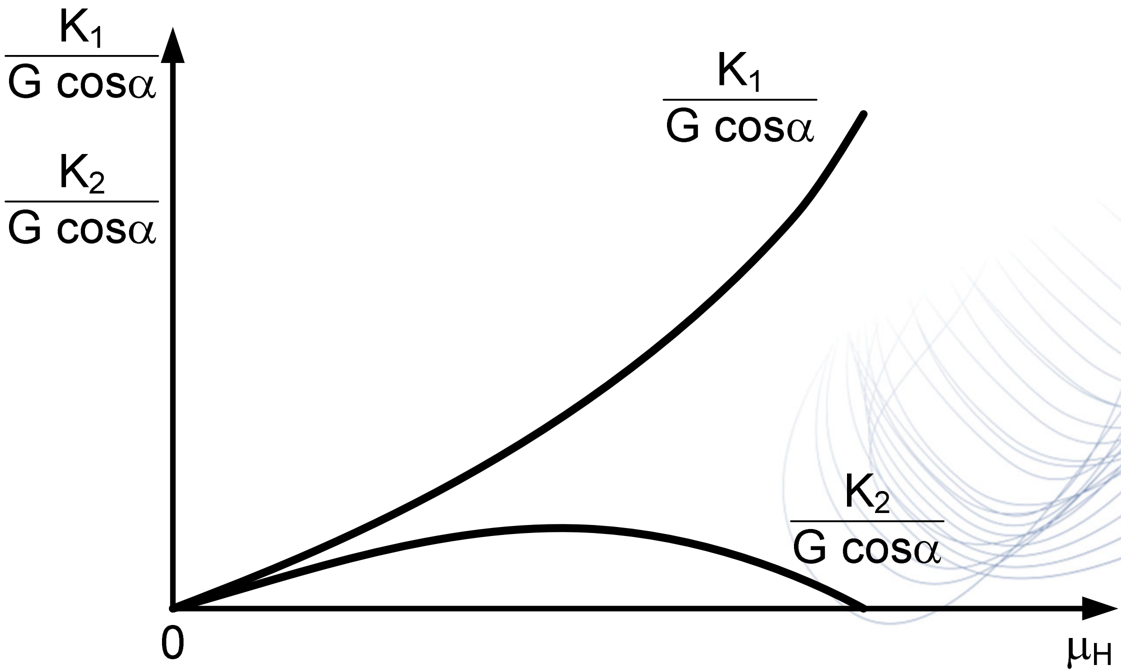
$$\frac{Z_1}{G \cdot \cos\alpha} = \frac{l''}{l} + \frac{h_n}{l} \cdot (f + \mu_H)$$

$$\frac{Z_2}{G \cdot \cos\alpha} = \frac{l'}{l} - \frac{h_n}{l} \cdot (f + \mu_H)$$

Maximal braking forces

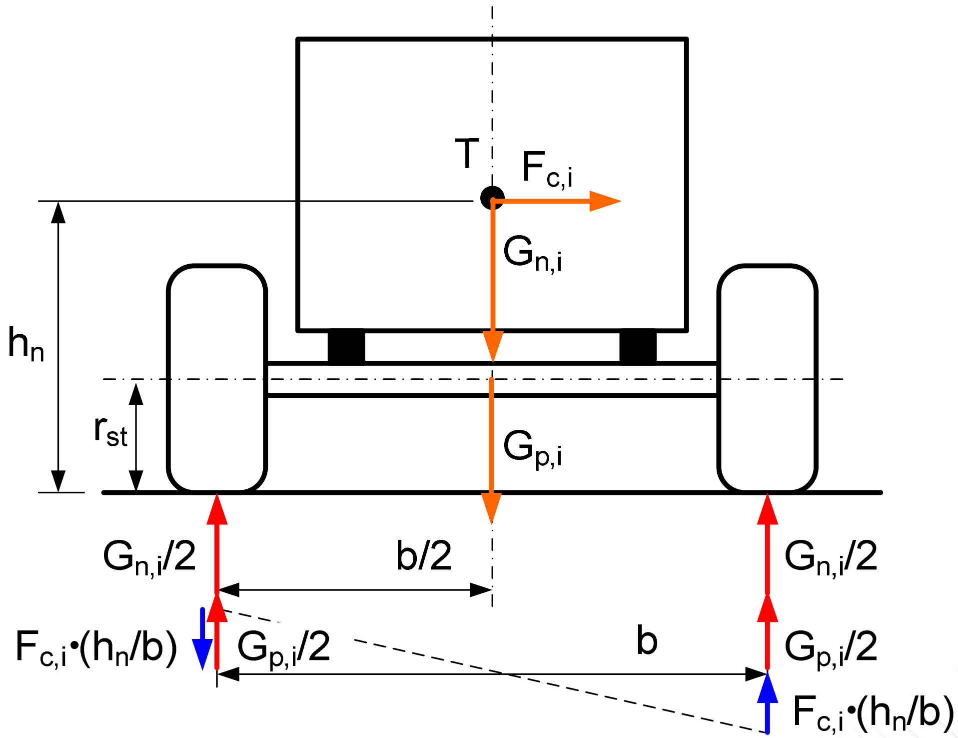
$$\frac{K_1}{G \cdot \cos\alpha} = \frac{\mu_H \cdot Z_1}{G \cdot \cos\alpha} = \mu_H \cdot \left(\frac{l''}{l} + \frac{h_n}{l} \cdot f \right) + \mu_H^2 \cdot \frac{h_n}{l}$$

$$\frac{K_2}{G \cdot \cos\alpha} = \frac{\mu_H \cdot Z_2}{G \cdot \cos\alpha} = \mu_H \cdot \left(\frac{l'}{l} - \frac{h_n}{l} \cdot f \right) - \mu_H^2 \cdot \frac{h_n}{l}$$



Vertical axle force-balance during cornering

No suspension between a chassis and an axle:



$$G_{n,1} = G \cdot \frac{l''}{l}; G_{n,2} = G \cdot \frac{l'}{l}$$

$$F_c = m_{n+p} \cdot \frac{v^2}{R_{ov}}; F_{c,1} = F_c \cdot \frac{l''}{l}; F_{c,2} = F_c \cdot \frac{l'}{l}$$

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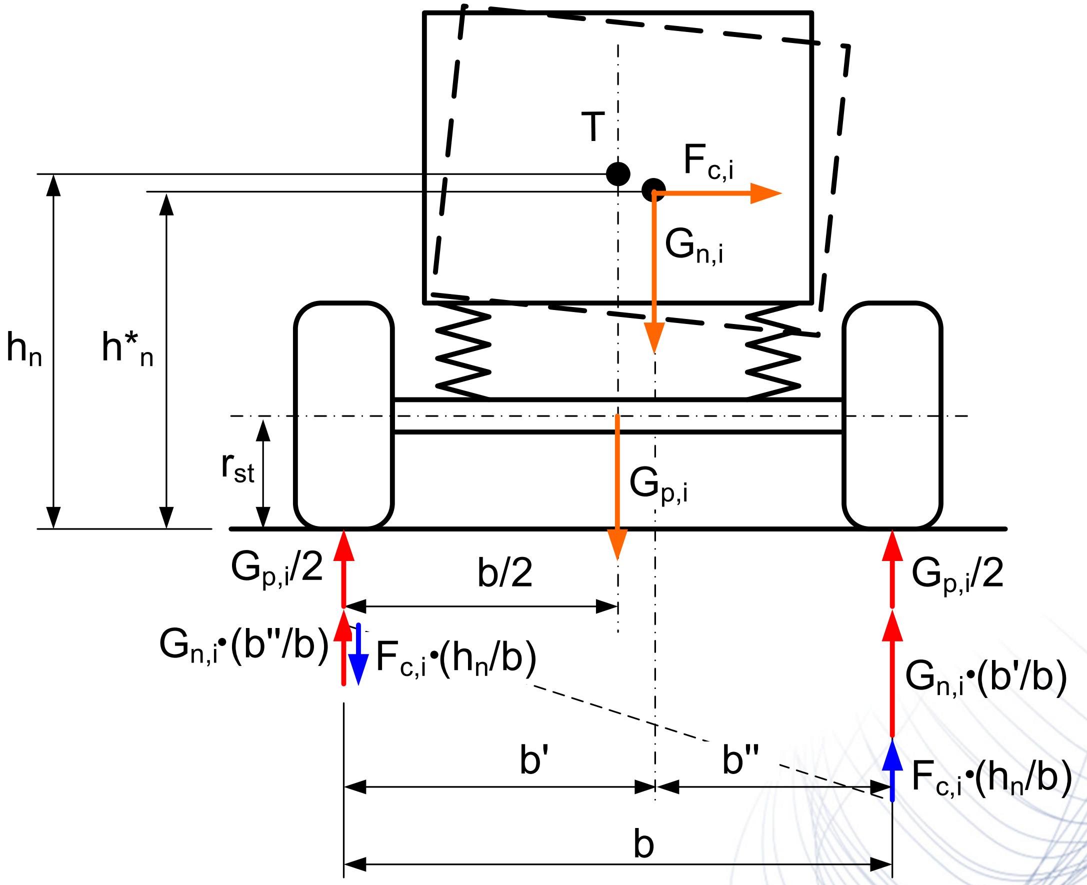
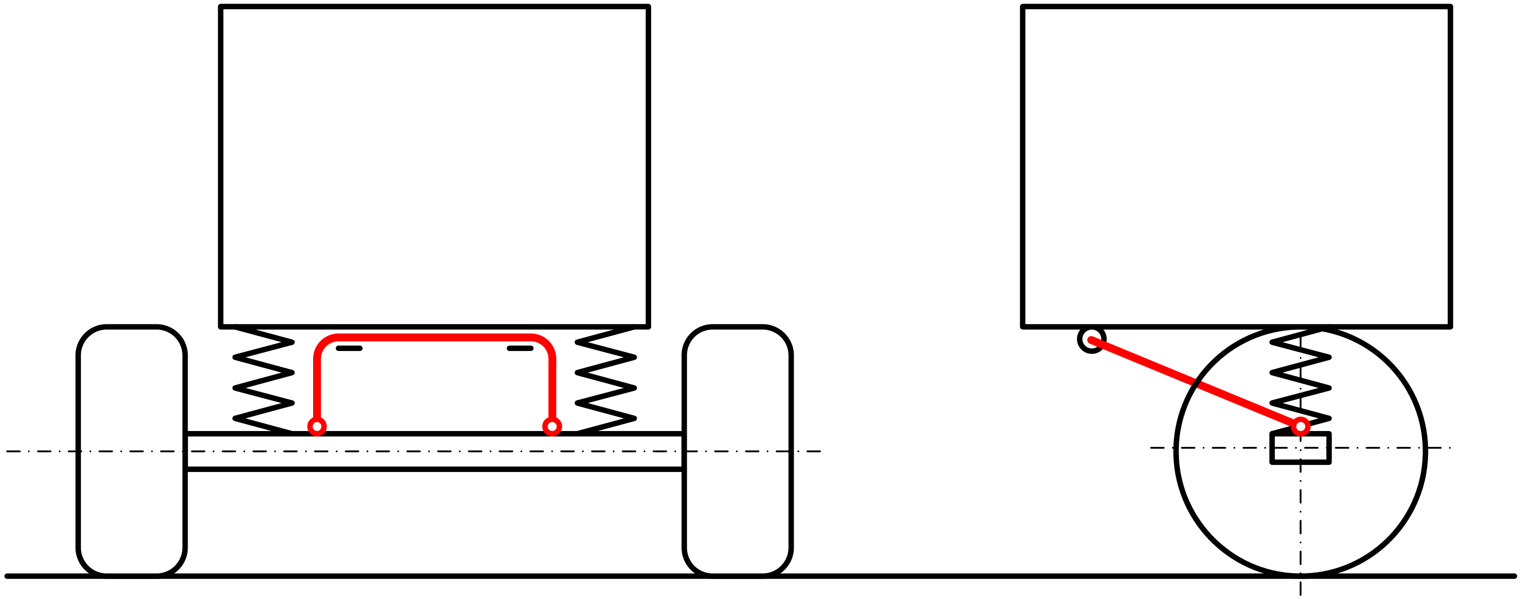
Vertical axle force-balance during cornering

With suspension between the chassis and the axle:



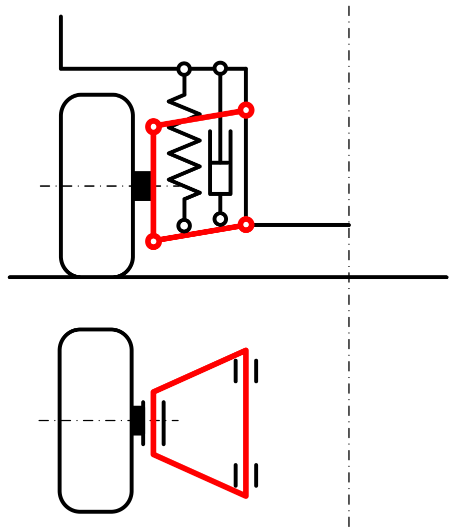
$$h_n^* \approx h_n ; b' \neq b'' \neq b/2$$

Axle torsion stabilizer

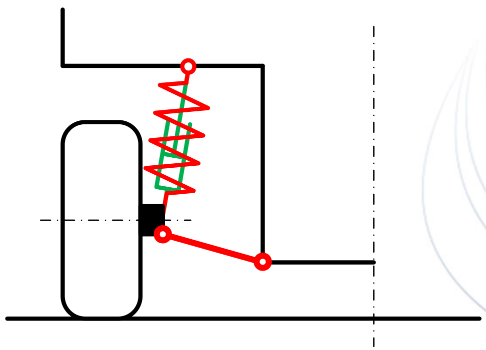


Wheel-suspension types

- *Independent suspension with double wishbone arms*:

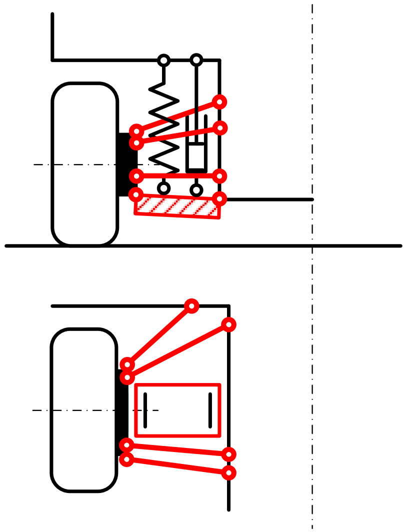


- *Independent suspension with McPherson strut*:

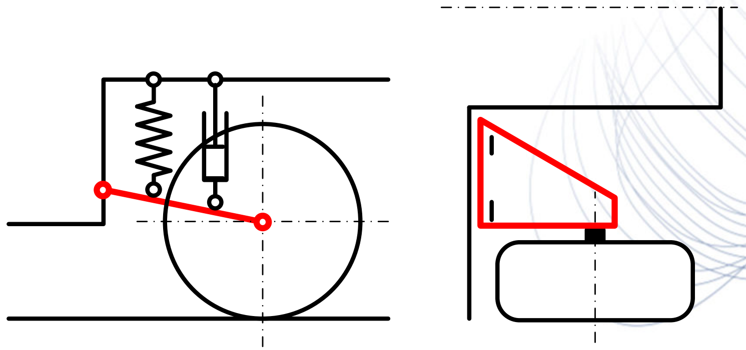


Wheel-suspension types

- *“multi-link“ suspension with a transverse trailing arm*:

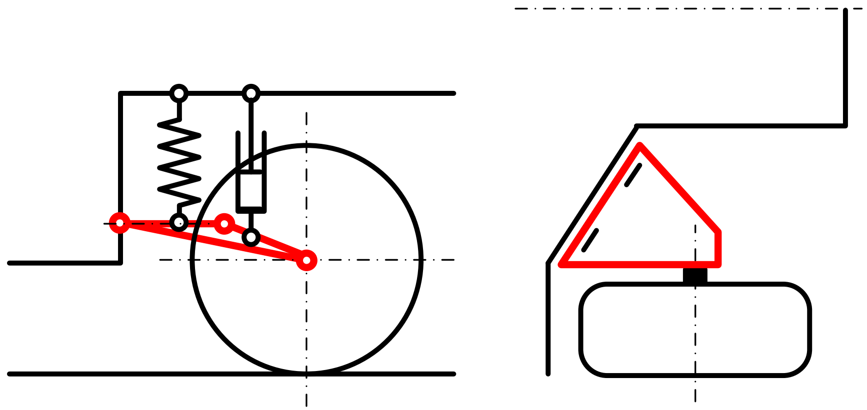


- *Independent suspension with a longitudinal trailing arm*:

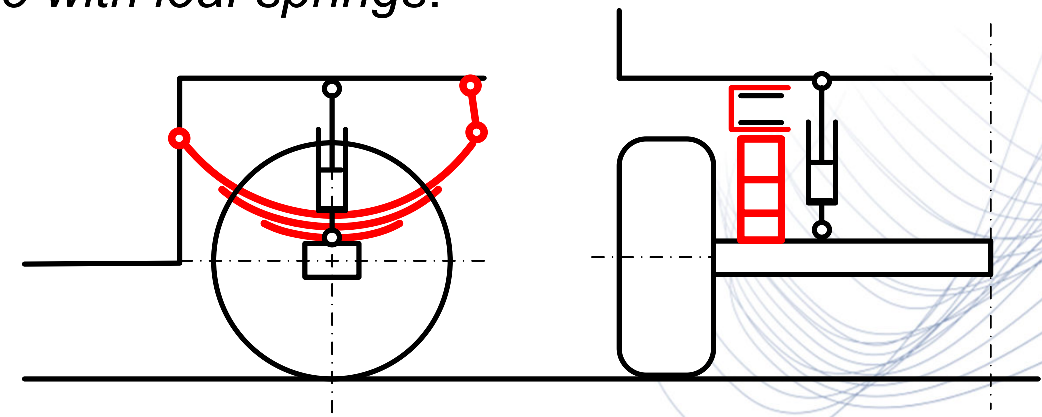


Wheel-suspension types

- *Independent suspension with a space-oriented trailing arm*:

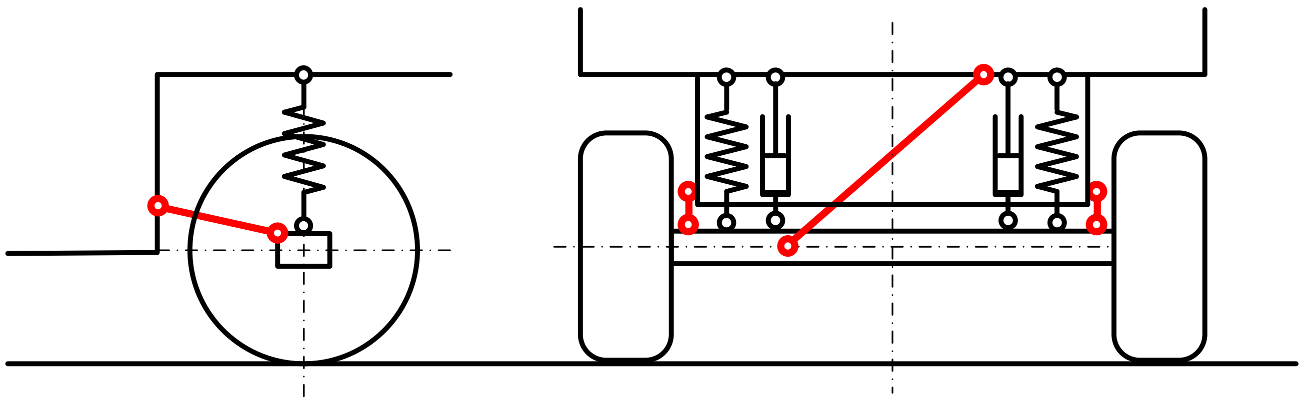


- *Rigid axle with leaf springs*:

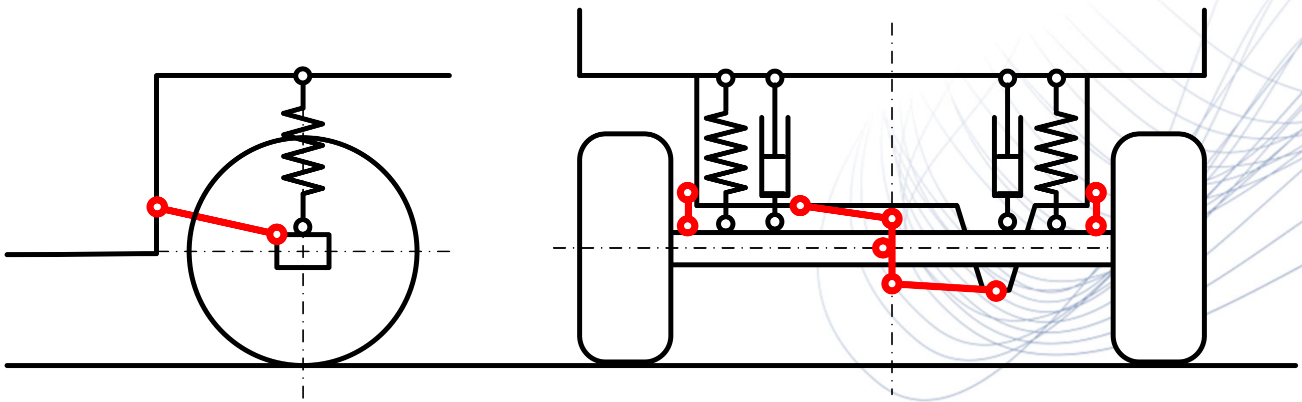


Wheel-suspension types

- *Rigid axle with a Panhard-rod*:

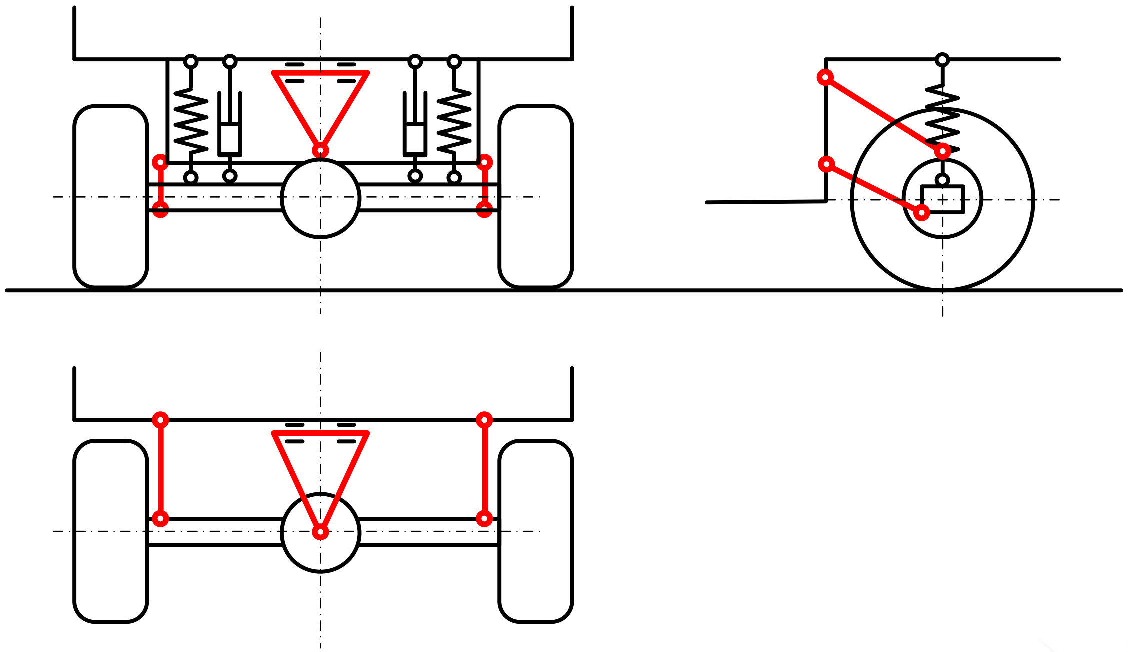


- *Rigid axle with a Watt's mechanism*:

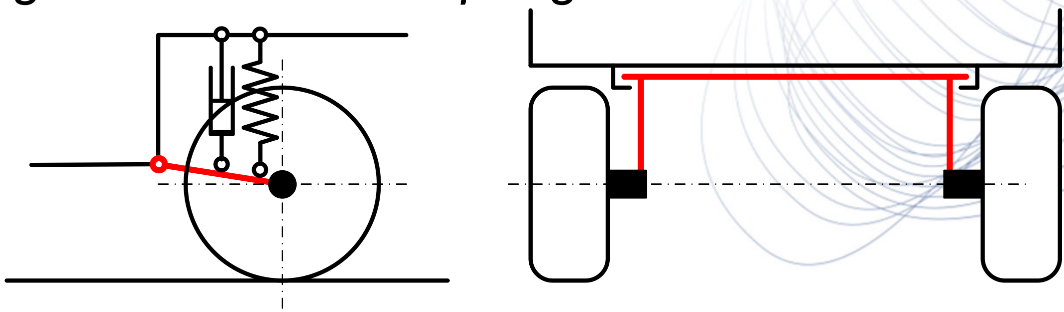


Wheel-suspension types

- *Rigid axle with longitudinal links*:



- *“Half-rigid“ axle with coil springs*:



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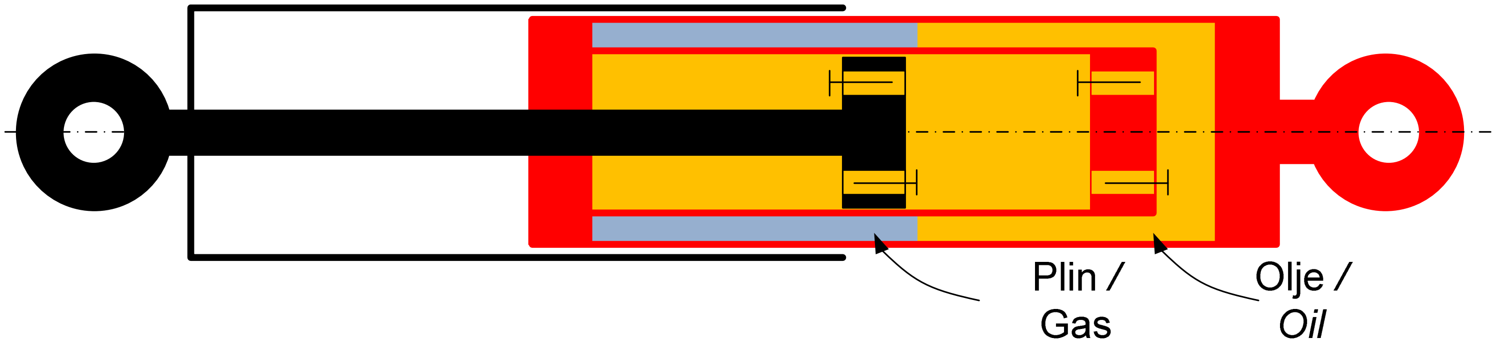
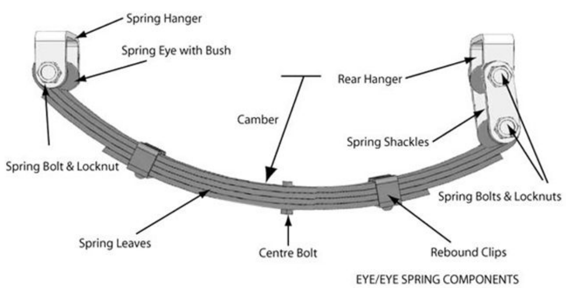
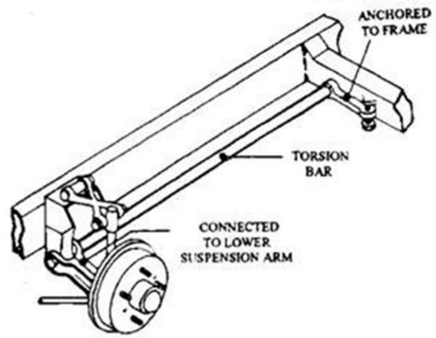
Wheel-suspension types

- *Double-chamber oil shock-absorber:*



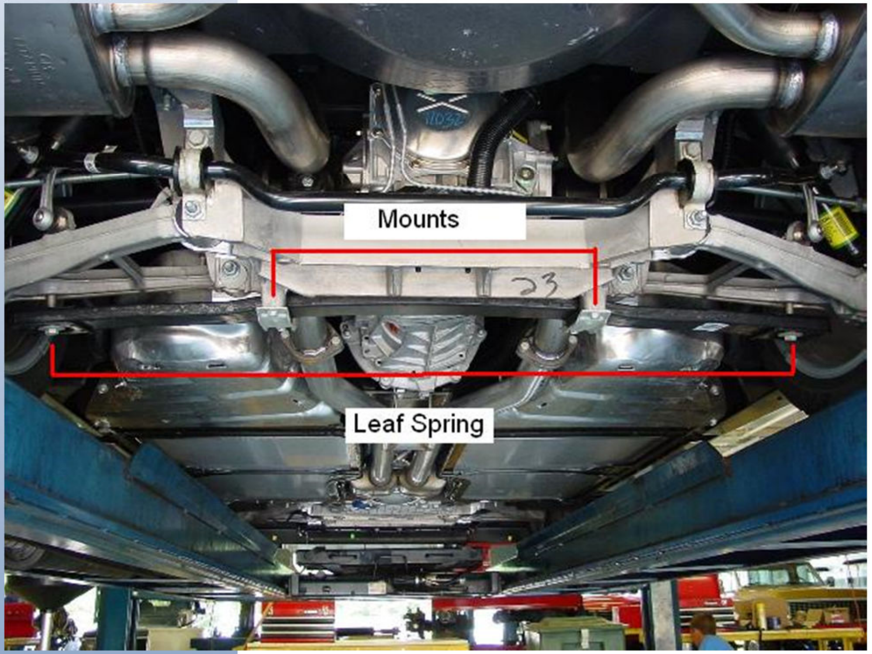
- *Steel springs:*
  - *A package of leaf springs (construction vehicles);*
  - *Torsion bar spring (Renault 4, 5-I);*
  - *Torsion coil spring (the most common suspension type of modern vehicles).*



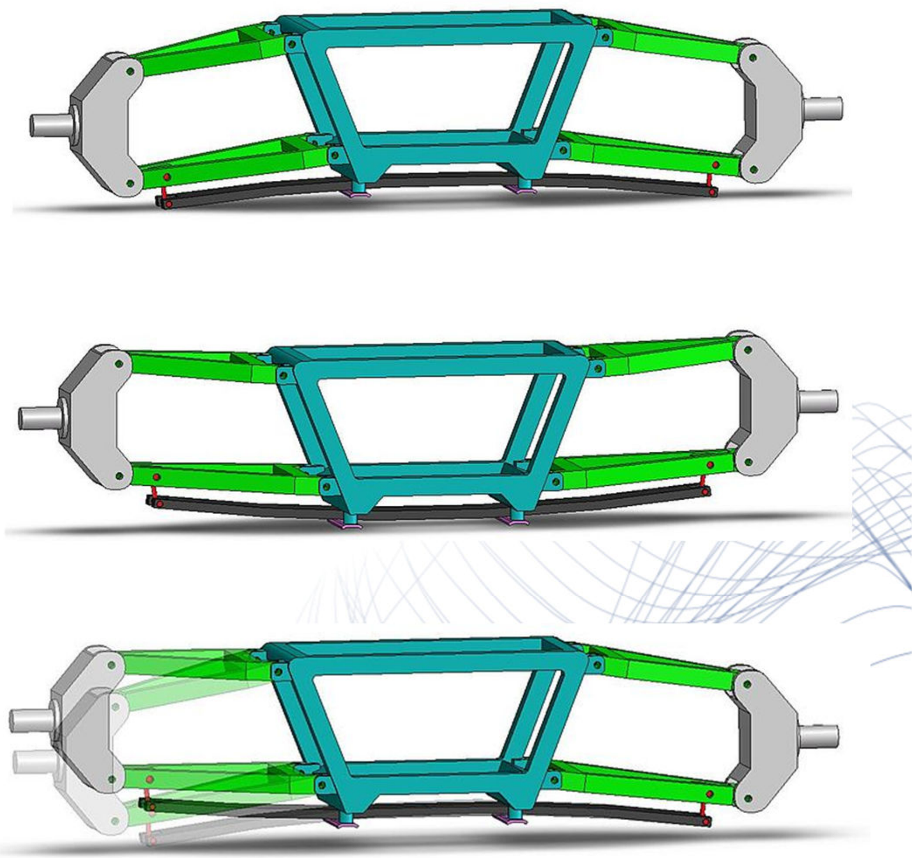


Wheel-suspension types

- *Composite bending spring (epoxy resin + glass fibers) – Chevrolete Corvette:*

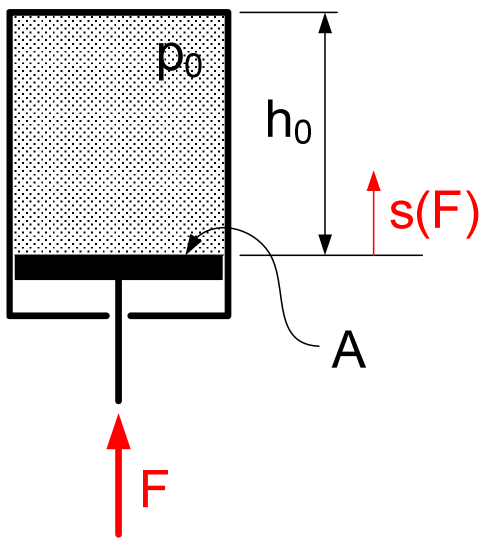


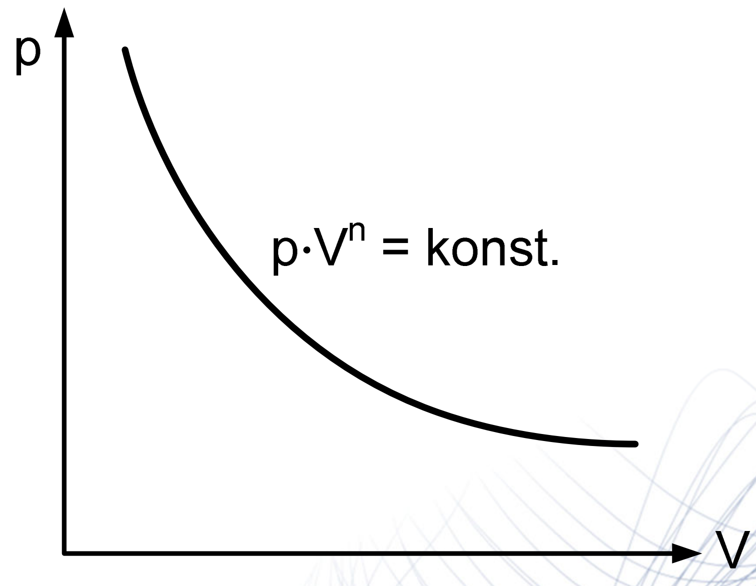
Vir: en.wikipedia.org/wiki/Corvette_leaf_spring



Wheel-suspension types

- *Pneumatic and hydro-pneumatic suspension – spring characteristics of a pneumatic spring:*





$n = 1{,}4$ ➡ isentropic curve

$n \approx 1{,}3$ ➡ polytrophic curve

Wheel-suspension types

- *Pneumatic and hydro-pneumatic suspension – spring characteristics of the pneumatic spring*:

$$p_0 \cdot V_0^n = p \cdot V^n$$

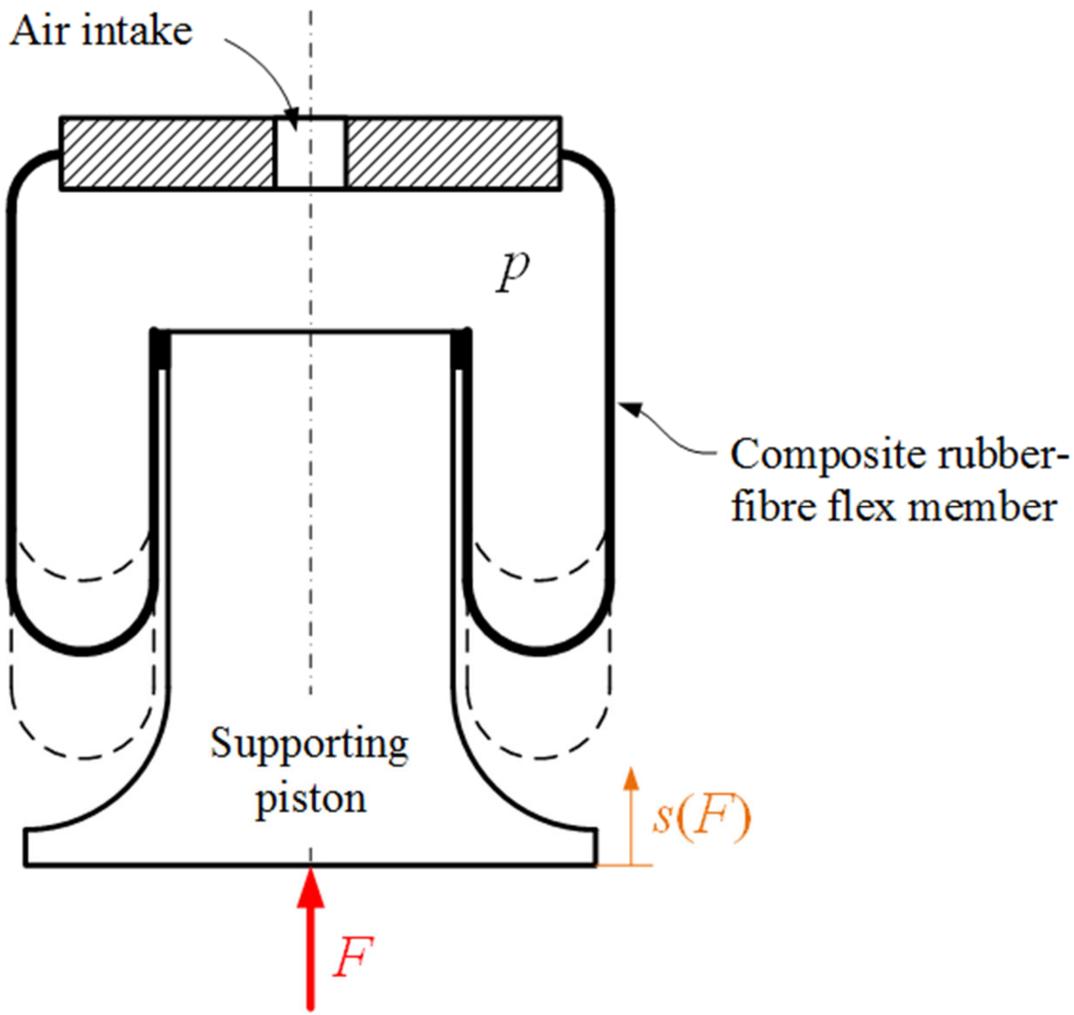
$$p_0 \cdot A^n \cdot h_0^n = \frac{F}{A} \cdot A^n \cdot (h_0 - s)^n$$

$$F = \frac{p_0 \cdot A \cdot h_0^n}{(h_0 - s)^n}$$

$$c = \frac{dF}{ds} = \frac{n \cdot p_0 \cdot A \cdot h_0^n}{(h_0 - s)^{n+1}}$$

Wheel-suspension types

- Pnevmatska vzmet z odvaljnim batom / *Pneumatic spring with a piston support*:



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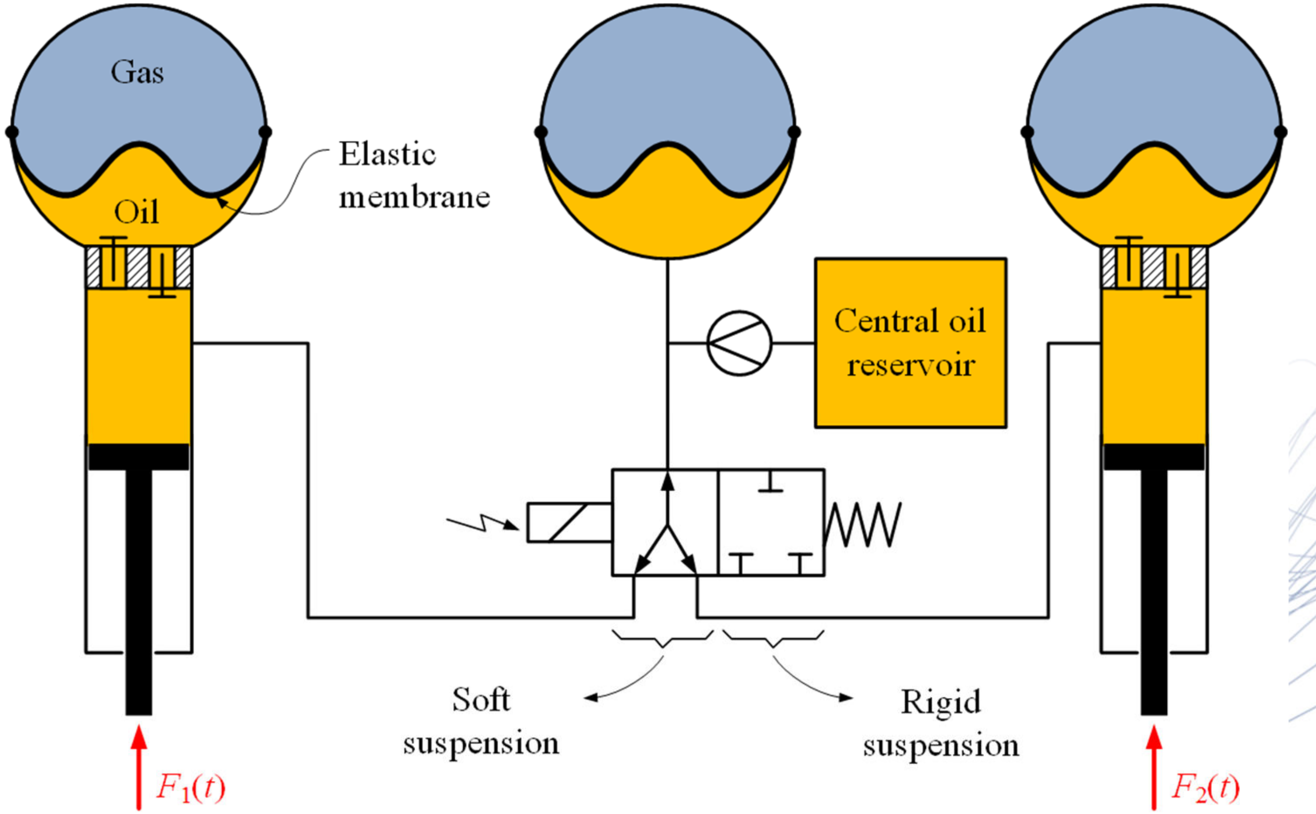
Wheel-suspension types

- *Citröen hydro-pneumatic suspension system (one axle):*

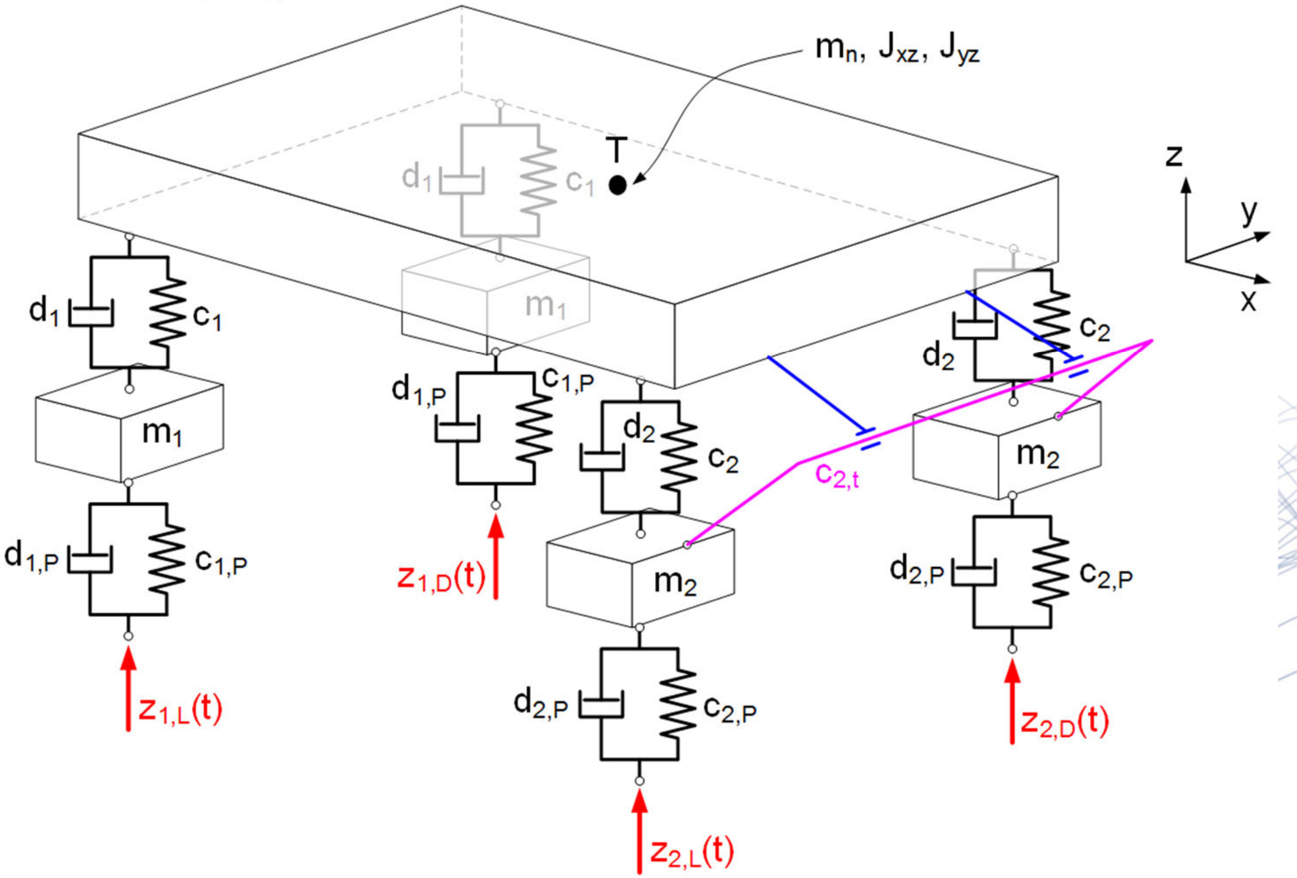


Vertical vibrations of suspended elements

Vehicle with independent suspension:

Degrees of freedom:

- *Vertical diplacements of wheels*: $z_{k1,L}$, $z_{k1,D}$, $z_{k2,L}$, $z_{k2,D}$
- *Vertical diplacements of chassis*: z_n
- *Rotations of chassis*: ϕ_{xz}, ϕ_{yz}

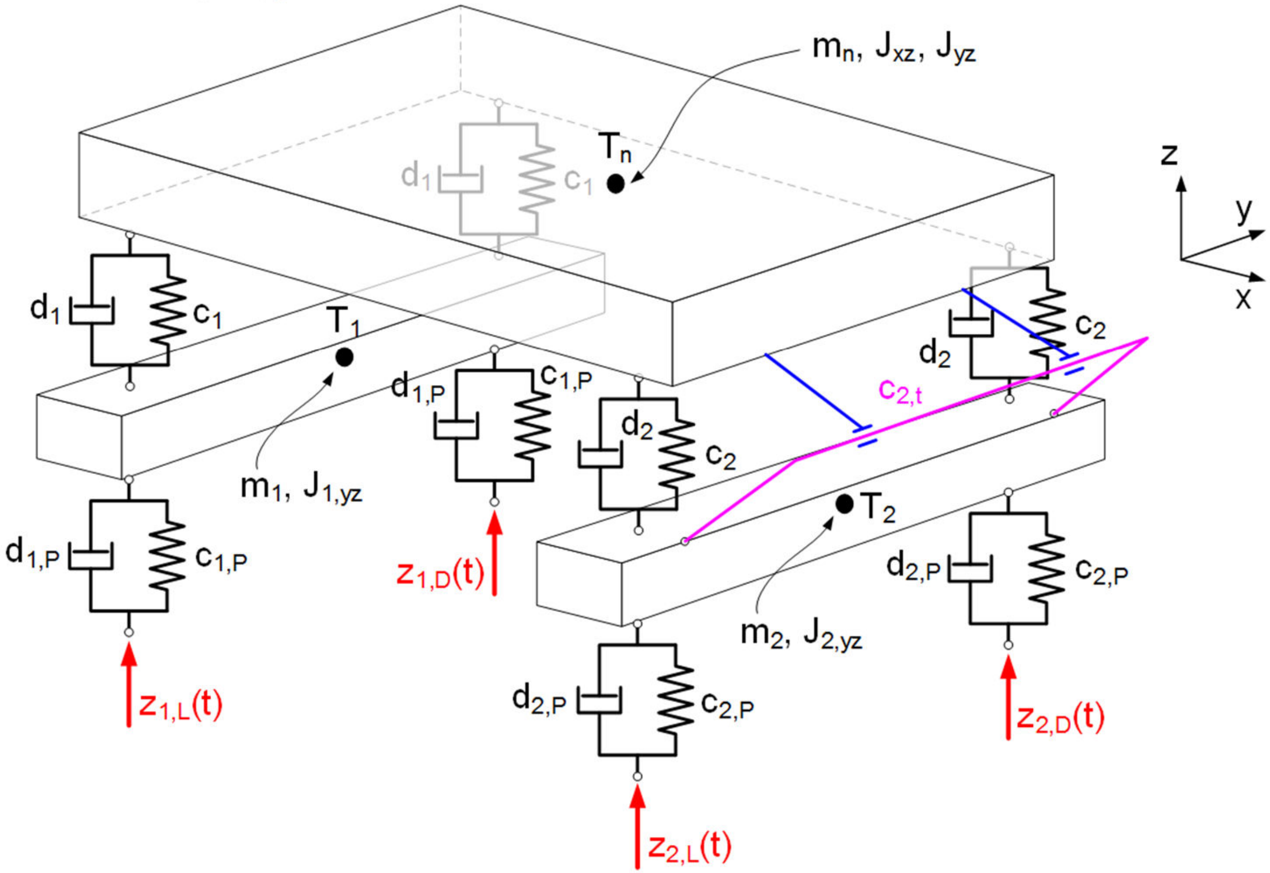


Vertical vibrations of suspended elements

Vehicle with rigid axles:

Degrees of freedom:

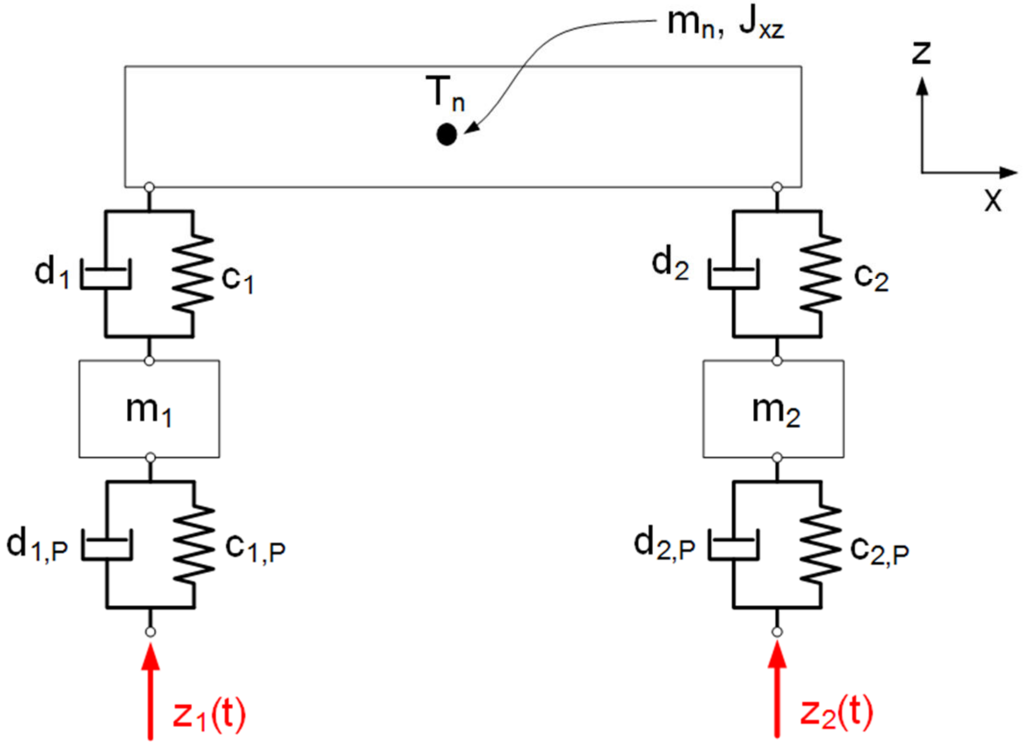
- *Vertical diplacements of wheels*: $z_{k1,L}$, $z_{k1,D}$, $z_{k2,L}$, $z_{k2,D}$
- *Vertical diplacements of chassis*: z_n
- *Rotations of chassis*: ϕ_{xz}, ϕ_{yz}



Vertical vibrations of suspended elements

Planar vehicle model:

Degrees of freedom:
- *Vertical diplacements of axles*: z_{k1}, z_{k2}
- *Vertical displacement and rotation of chassis*: z_n, ϕ_{xz}



Vertical vibrations of suspended elements

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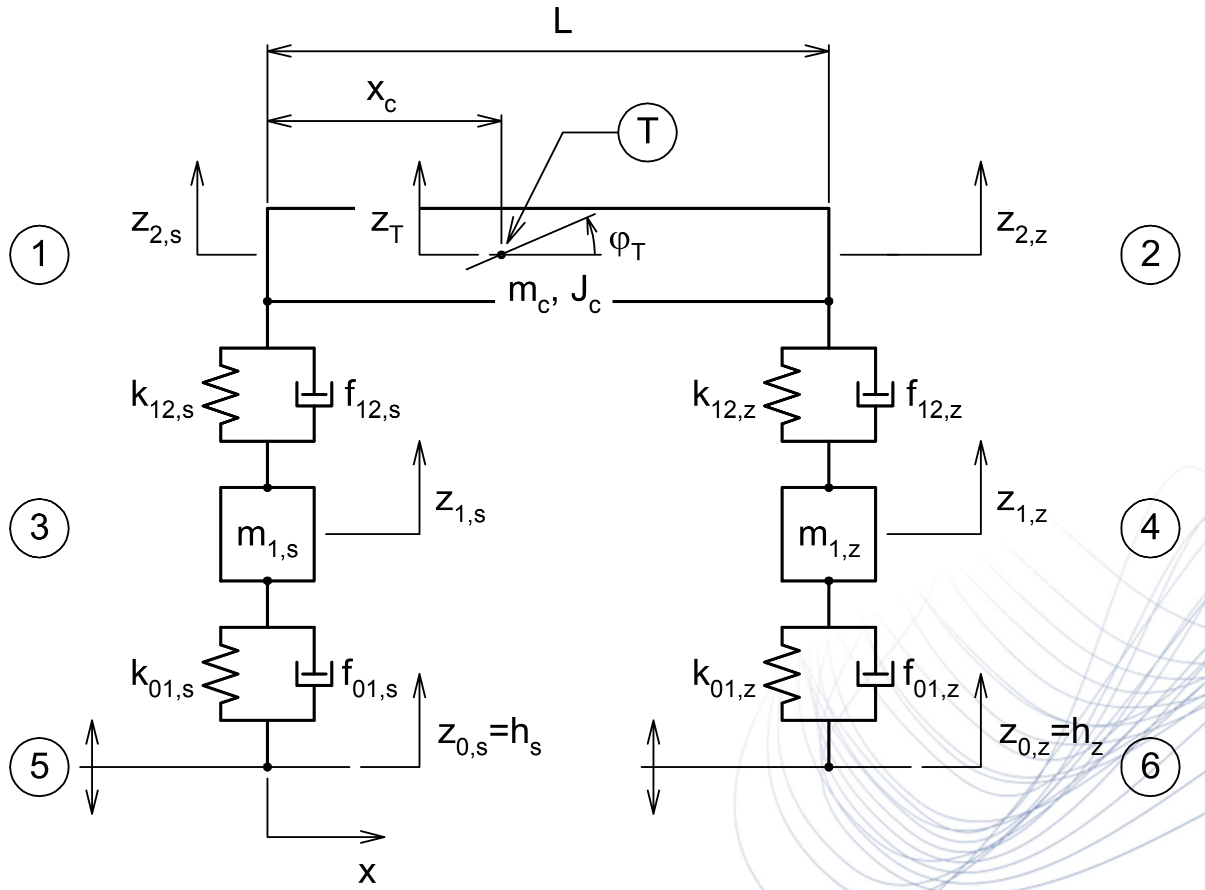
An example of governing equations for the planar model of the vehicle:



Vertical vibrations of suspended elements

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- *Presumptions*:

$$z_{2,s} > z_{1,s} > h_s$$

$$z_{2,z} > z_{1,z} > h_z$$

$$z_{2,z} > z_{2,s}$$

$$\dot{z}_{2,s} > \dot{z}_{1,s} > \dot{h}_s$$

$$\dot{z}_{2,z} > \dot{z}_{1,z} > \dot{h}_z$$

$$z_T = z_{2,s} \cdot \left(1 - \frac{x_c}{L}\right) + z_{2,z} \cdot \frac{x_c}{L}$$

$$\varphi_T \approx \tan\varphi_T = \frac{z_{2,z} - z_{2,s}}{L}$$

$$z_{2,s} = z_T - \varphi_T \cdot x_c$$

$$z_{2,z} = z_T + \varphi_T \cdot (L - x_c)$$

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Vertical vibrations of suspended elements

- *Governing equations*:

$$m_{1,s} \cdot \ddot{z}_{1,s} = k_{12,s}\left(z_T - \varphi_T \cdot x_c - z_{1,s}\right) + f_{12,s}\left(\dot{z}_T - \dot{\varphi}_T \cdot x_c - \dot{z}_{1,s}\right) - \\ - k_{01,s}\left(z_{1,s} - h_s\right) - f_{01,s}\left(\dot{z}_{1,s} - \dot{h}_s\right) + F_{01,s} - F_{12,s} - m_{1,s} \cdot g$$

$$m_{1,z} \cdot \ddot{z}_{1,z} = k_{12,z}\left(z_T + \varphi_T\left(L - x_c\right) - z_{1,z}\right) + f_{12,z}\left(\dot{z}_T + \dot{\varphi}_T\left(L - x_c\right) - \dot{z}_{1,z}\right) - \\ - k_{01,z}\left(z_{1,z} - h_z\right) - f_{01,z}\left(\dot{z}_{1,z} - \dot{h}_z\right) + F_{01,z} - F_{12,z} - m_{1,z} \cdot g$$

$$m_c \cdot \ddot{z}_T = -k_{12,s}\left(z_T - \varphi_T \cdot x_c - z_{1,s}\right) - f_{12,s}\left(\dot{z}_T - \dot{\varphi}_T \cdot x_c - \dot{z}_{1,s}\right) - \\ - k_{12,z}\left(z_T + \varphi_T\left(L - x_c\right) - z_{1,z}\right) - f_{12,z}\left(\dot{z}_T + \dot{\varphi}_T\left(L - x_c\right) - \dot{z}_{1,z}\right) + \\ + F_{12,s} + F_{12,z} - m_c \cdot g$$

$$J_c \cdot \ddot{\varphi}_T = \left[k_{12,s}\left(z_T - \varphi_T \cdot x_c - z_{1,s}\right) + f_{12,s}\left(\dot{z}_T - \dot{\varphi}_T \cdot x_c - \dot{z}_{1,s}\right)\right] \cdot x_c - \\ - \left[k_{12,z}\left(z_T + \varphi_T\left(L - x_c\right) - z_{1,z}\right) + f_{12,z}\left(\dot{z}_T + \dot{\varphi}_T\left(L - x_c\right) - \dot{z}_{1,z}\right)\right] \cdot \left(L - x_c\right) - \\ - F_{12,s} \cdot x_c + F_{12,z}\left(L - x_c\right)$$

Vertical vibrations of suspended elements

- *System of differential equations of the 1st order for local coordinate systems (*) and matrix formulation:*

$$z_{1,s} = y_1 + z_1^* \Rightarrow \dot{y}_1 = \dot{z}_{1,s}\,, \ddot{y}_1 = \ddot{z}_{1,s} \qquad y_5 = \dot{y}_1$$

$$z_{1,z} = y_2 + z_2^* \Rightarrow \dot{y}_2 = \dot{z}_{1,z}\,, \ddot{y}_2 = \ddot{z}_{1,z} \qquad y_6 = \dot{y}_2$$

$$z_T = y_3 + z_3^* \Rightarrow \dot{y}_3 = \dot{z}_T\,, \ddot{y}_3 = \ddot{z}_T \qquad y_7 = \dot{y}_3$$

$$\varphi_T = y_4 + z_4^* \Rightarrow \dot{y}_4 = \dot{\varphi}_T\,, \ddot{y}_4 = \ddot{\varphi}_T \qquad y_8 = \dot{y}_4$$

$$z_{0,s} = h_s = y' + z' \Rightarrow \dot{y}' = \dot{h}_s$$

$$z_{0,z} = h_z = y'' + z'' \Rightarrow \dot{y}'' = \dot{h}_z$$

$$\dot{\mathbf{y}} = \mathbf{A} \cdot \mathbf{y} + \mathbf{u}_G + \mathbf{u}_F + \mathbf{u}_P + \mathbf{u}_K$$

$$\mathbf{y} = \left(y_1, y_2, y_3, y_4, y_5, y_6, y_7, y_8\right)^T$$

$$\dot{\mathbf{y}} = \left(\dot{y}_1, \dot{y}_2, \dot{y}_3, \dot{y}_4, \dot{y}_5, \dot{y}_6, \dot{y}_7, \dot{y}_8\right)^T$$

Vertical vibrations of suspended elements

$$\mathbf{A}=\begin{bmatrix}\mathbf{0}\ ,\ \mathbf{I}_4\\ \mathbf{K}\ ,\mathbf{F}\end{bmatrix}$$

$$\mathbf{K}=\begin{bmatrix}\left(-k_{12,s}-k_{01,s}\right)/m_{1,s} & 0 & k_{12,s}/m_{1,s} & -k_{12,s}x_c/m_{1,s}\\ 0 & \left(-k_{12,z}-k_{01,z}\right)/m_{1,z} & k_{12,z}/m_{1,z} & -k_{12,z}\left(L-x_c\right)/m_{1,z}\\ k_{12,s}/m_c & k_{12,z}/m_c & \left(-k_{12,s}-k_{12,z}\right)/m_c & \left(k_{12,s}x_c-k_{12,z}\left(L-x_c\right)\right)/m_c\\ -k_{12,s}x_c/J_c & k_{12,z}\left(L-x_c\right)/J_c & \left(k_{12,s}x_c-k_{12,z}\left(L-x_c\right)\right)/J_c & \left(-k_{12,s}x_c^2-k_{12,z}\left(L-x_c\right)^2\right)/J_c\end{bmatrix}$$

$$\mathbf{F}=\begin{bmatrix}\left(-f_{12,s}-f_{01,s}\right)/m_{1,s} & 0 & f_{12,s}/m_{1,s} & -f_{12,s}x_c/m_{1,s}\\ 0 & \left(-f_{12,z}-f_{01,z}\right)/m_{1,z} & f_{12,z}/m_{1,z} & -f_{12,z}\left(L-x_c\right)/m_{1,z}\\ f_{12,s}/m_c & f_{12,z}/m_c & \left(-f_{12,s}-f_{12,z}\right)/m_c & \left(f_{12,s}x_c-f_{12,z}\left(L-x_c\right)\right)/m_c\\ -f_{12,s}x_c/J_c & f_{12,z}\left(L-x_c\right)/J_c & \left(f_{12,s}x_c-f_{12,z}\left(L-x_c\right)\right)/J_c & \left(-f_{12,s}x_c^2-f_{12,z}\left(L-x_c\right)^2\right)/J_c\end{bmatrix}$$

$$\mathbf{u}_G=\left(0,0,0,0,-g,-g,-g,0\right)^T$$

Vertical vibrations of suspended elements

$$\mathbf{u}_F = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \left(F_{01,s} - F_{12,s}\right)/m_{1,s} \\ \left(F_{01,z} - F_{12,z}\right)/m_{1,z} \\ \left(F_{12,s} + F_{12,z}\right)/m_c \\ \left(F_{12,z} \cdot (L - x_c) - F_{12,s} \cdot x_c\right)/J_c \end{bmatrix} \qquad \mathbf{u}_P = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \left[k_{01,s} \cdot y' + f_{01,s} \cdot \dot{y}'\right]/m_{1,s} \\ \left[k_{01,z} \cdot y'' + f_{01,z} \cdot \dot{y}''\right]/m_{1,s} \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{u}_K = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ \left[k_{12,s} \cdot \left(-z_1^* + z_3^* - z_4^* \cdot x_c\right) + k_{01,s} \cdot \left(z' - z_1^*\right)\right]/m_{1,s} \\ \left[k_{12,z} \cdot \left(-z_2^* + z_3^* + z_4^* \cdot (L - x_c)\right) + k_{01,z} \cdot \left(z'' - z_2^*\right)\right]/m_{1,z} \\ \left[k_{12,s} \cdot \left(z_1^* - z_3^* + z_4^* \cdot x_c\right) + k_{12,z} \cdot \left(z_2^* - z_3^* - z_4^* \cdot (L - x_c)\right)\right]/m_c \\ \left[k_{12,s} \cdot \left(z_1^* - z_3^* + z_4^* \cdot x_c\right) \cdot x_c - k_{12,z} \cdot \left(z_2^* - z_3^* - z_4^* \cdot (L - x_c)\right) \cdot (L - x_c)\right]/J_c \end{bmatrix}$$

Vertical vibrations of suspended elements

- *Numerical solution of the system of differential equations with an implicit derivation according to Euler:*

$$\dot{\mathbf{y}}(n+1) = \frac{\mathbf{y}(n+1) - \mathbf{y}(n)}{t(n+1) - t(n)}$$

$$\mathbf{y}(n+1) = \mathbf{y}(n) + \big(t(n+1) - t(n)\big) \cdot \dot{\mathbf{y}}(n+1)$$

$$\mathbf{y}(n+1) = \mathbf{y}(n) + \big(t(n+1) - t(n)\big) \cdot$$
$$\cdot \big[\mathbf{A} \cdot \mathbf{y}(n+1) + \mathbf{u}_G(n+1) + \mathbf{u}_F(n+1) + \mathbf{u}_P(n+1) + \mathbf{u}_K(n+1)\big]$$